Mathematics: analysis and approaches
Higher level
Paper 1

30 October 2023

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

15 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Consider the functions $f(x) = x - 3$ and $g(x) = x^2 + k^2$, where k is a real constant.

(a) Write down an expression for $(g \circ f)(x)$. [2]

(b) Given that $(g \circ f)(2) = 10$, find the possible values of k. [3]

2. [Maximum mark: 4]

Events A and B are such that $\mathrm{P}(A) = 0.65$, $\mathrm{P}(B) = 0.75$ and $\mathrm{P}(A \cap B) = 0.6$.

(a) Find $\mathrm{P}(A \cup B)$. [2]

(b) Hence, or otherwise, find $\mathrm{P}(A' \cap B')$. [2]

3. [Maximum mark: 7]

The sum of the first n terms of an arithmetic sequence is given by $S_n = pn^2 - qn$, where p and q are positive constants.

It is given that $S_4 = 40$ and $S_5 = 65$.

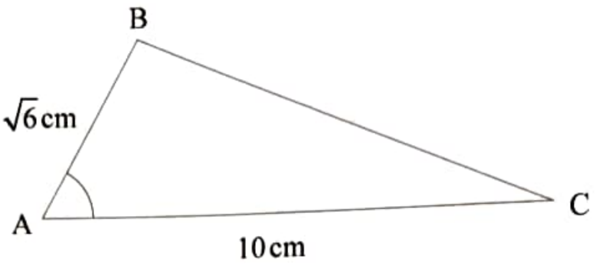
(a) Find the value of p and the value of q. [5]

(b) Find the value of u_5. [2]

4. [Maximum mark: 6]

In the following triangle ABC, $\mathrm{AB} = \sqrt{6}\ \mathrm{cm}$, $\mathrm{AC} = 10\,\mathrm{cm}$ and $\cos \mathrm{B\hat{A}C} = \frac{1}{5}$.

diagram not to scale



Find the area of triangle ABC.

5. [Maximum mark: 6]

The binomial expansion of $(1 + kx)^n$ is given by $1 + 12x + 28k^2x^2 + \ldots + k^nx^n$ where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of n and the value of k.

6. [Maximum mark: 7]

Prove by mathematical induction that $5^{2n} - 2^{3n}$ is divisible by 17 for all $n \in \mathbb{Z}^{+}$.

7. [Maximum mark: 5]

It is given that $z = 5 + q\mathrm{i}$ satisfies the equation $z^2 + \mathrm{i}z = -p + 25\mathrm{i}$, where $p, q \in \mathbb{R}$.

Find the value of p and the value of q.

8. [Maximum mark: 9]

(a) Find $\int x(\ln x)^2 \mathrm{d}x$. [6]

(b) Hence, show that $\int_1^4 x(\ln x)^2 \mathrm{d}x = 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$. [3]

9. [Maximum mark: 8]

Consider the function $f(x)=\frac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that f is an even function. [2]

(b) Given that $\lim_{x\to 0} f(x)=16$, find the value of k. [6]

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The functions f and g are defined by

$f(x) = \ln(2x - 9)$, where $x > \frac{9}{2}$

$g(x) = 2\ln x - \ln d$, where $x > 0$, $d \in \mathbb{R}^+$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b) (i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$.

(ii) Hence show that $d^2 - 9d > 0$.

(iii) Find the range of possible values of d. [9]

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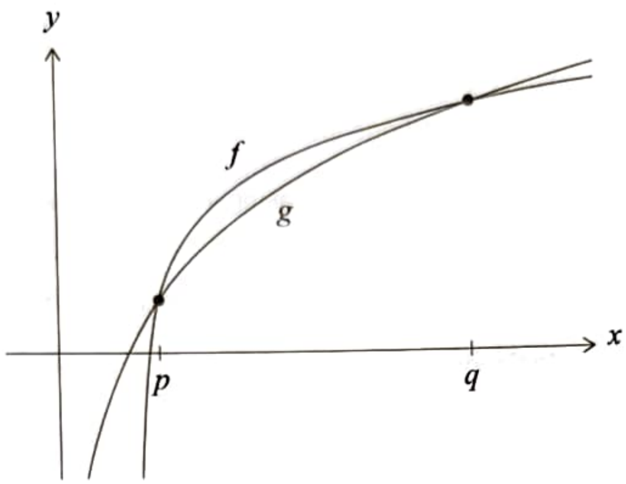
(Question 10 continued)

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

(c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where $a, b \in \mathbb{Z}^+$. [5]

Do **not** write solutions on this page.

11. [Maximum mark: 21]

Consider the function $f(x) = \mathrm{e}^{\cos 2x}$, where $-\frac{\pi}{4} \le x \le \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

(c) Sketch the curve of $y = f(x)$ for $0 \le x \le \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

(d) (i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4.

(ii) Hence, find the Maclaurin series for $\mathrm{e}^{\cos 2x - 1}$, up to and including the term in x^4.

(iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4. [6]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} \mathrm{e}^{\cos 2x}\,\mathrm{d}x \approx \frac{149\mathrm{e}}{1500}$. [3]

Do **not** write solutions on this page.

12. [Maximum mark: 17]

(a) Find the binomial expansion of $(\cos\theta + \mathrm{i}\sin\theta)^5$. Give your answer in the form $a + b\mathrm{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. **[4]**

(b) By using De Moivre's theorem and your answer to part (a), show that $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. **[6]**

(c) (i) Hence, show that $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solutions of the equation $16\sin^4\theta - 20\sin^2\theta + 5 = 0$.

(ii) Hence, show that $\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$. **[7]**